Contents

Chapter 10

System of Linear Differential Equations

10.1 Theory of Linear System

We start from an example.

Example 10.1.1. Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and consider the system of DE

$$\begin{array}{rcl} \frac{dx}{dt} &=& 2x+3y \\ \frac{dy}{dt} &=& -4x+5y \end{array} \quad \text{or} \quad \mathbf{x}' = \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix} \mathbf{x}.$$

Example 10.1.2. Verification of solutions: The vector functions

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

are solutions of the DE.

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{x}. \tag{10.1}$$

More generally, we consider the first order system of linear differential equation in n-unknowns given by

$$\begin{array}{rcl} x_1' &=& a_{11}(t)x_1 + \cdots a_{1n}(t)x_n + f_1(t) \\ x_2' &=& a_{21}(t)x_1 + \cdots a_{2n}(t)x_n + f_2(t) \\ \cdot & \cdot & \cdots \\ x_n' &=& a_{n1}(t)x_1 + \cdots a_{nn}(t)x_n + f_n(t) \end{array} \tag{10.2}$$

In matrix form (10.2) becomes

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}, \tag{10.3}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{x}' = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}, \ A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Theorem 10.1.3. *[Existence and uniqueness] Assume $a_{11}(t), a_{12}(t), \cdots, a_{1n}(t), \cdots, a_{nn}(t), f_1(t), \cdots, f_n(t)$ are continuous on the interval $a < t < b$. Then for $a < t_0 < b$ the DE (10.2), or (10.3) has a unique solution satisfying ICs; $x_1(t_0) = x_1^0, \cdots, x_n(t_0) = x_n^0$.*

Consider the homogeneous case.

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{10.4}$$

Example 10.1.4. Consider the DE.

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

The solutions are

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t \\ -\frac{1}{2}(\cos t - \sin t) \\ -\cos t - \sin t \end{pmatrix} \text{ and } \mathbf{x}_2(t) = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

Hence

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} \cos t \\ -\frac{1}{2}(\cos t - \sin t) \\ -\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

is another solution of the homogeneous system. Acually, there is a third solution.

Linear dependence/independence

Definition 10.1.5. [Linear independence] If $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ are solutions of (10.4) in $a < t < b$, then we say the set of solution vectors are **linearly dependent** dependent if there exist constants $c_1, \cdots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)} = 0$$

holds for all $t \in (a, b)$. Otherwise, they are called **linearly independent**.

Given a set of solution vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} x_{12} \\ \vdots \\ x_{n2} \end{pmatrix}, \cdots, \mathbf{x}^{(n)} = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix},$$

the **Wronskian** W is defined as

$$W(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}) = \begin{vmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \cdots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{vmatrix}. \tag{10.5}$$

Theorem 10.1.6. *[Criterion for linear independence] If $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ are solutions of (10.4) then the set of solution vectors are linearly independent if and only if*

$$W(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}) \neq 0. \tag{10.6}$$

for every t in the interval.

Remark 10.1.7. To show the Wronskian is nonzero at all point, it suffices to show the Wronskian is nonzero at any one point.

Theorem 10.1.8. *[Superposition principle] If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are the solutions of (10.4) then for any constants $c_1, c_2, \cdots, c_n$ the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$ is also a solution of (10.4).*

Now study the general solution of (10.4).

Theorem 10.1.9. *[General solutions of system of homogenous DEs] If $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ are linear independent solutions of DE (10.4) in $a < t < b$, then any solution $\boldsymbol{\phi}(t)$ is given by a linear combination of $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$:*

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)} \tag{10.7}$$

Proof. Let $\boldsymbol{\phi}$ be any solution of (10.4). Fix a point $t_0 (a < t_0 < b)$, set $\boldsymbol{\phi}(t_0) = \mathbf{k} = (k_1, \cdots, k_n)$. Then for the general solution $c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$ to satisfy the ICs $\mathbf{x}(t_0) = \mathbf{k}$, i.e.,

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) = \mathbf{k}, \tag{10.8}$$

we must have

$$\begin{aligned} c_1 x_{11}(t_0) + \cdots + c_n x_{1n}(t_0) &= k_1, \\ &\cdots \\ c_1 x_{n1}(t_0) + \cdots + c_n x_{nn}(t_0) &= k_n. \end{aligned}$$

This is a system of linear equations in $c_1, \cdots, c_n$. By hypothesis, the functions are linearly independent, i.e.,

$$W(\mathbf{x}^{(1)}(t_0), \cdots, \mathbf{x}^{(n)}(t_0)) \neq 0. \tag{10.9}$$

Hence the solution $c_1, \cdots, c_n$ exists uniquely. Thus the solution of the IVP is $\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$. □

Definition 10.1.10. Any set $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ of n linearly independent solution vectors is said to be **fundamental set of solutions** of (10.4).

For simplicity we consider the case $t_0 = 0$ only.

Theorem 10.1.11. *Let* $\mathbf{x}^{(i)}, (i = 1, 2, \cdots, n)$ *be the solution of IVPs*

$$\begin{aligned} \mathbf{x}'(t) &= A(t)\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{e}^{(i)}. \end{aligned} \tag{10.10}$$

Then $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ *are the fundamental set of solutions. For any IC.* $\mathbf{x}(0) = \mathbf{k} = (k_1, \cdots, k_n)^T$*, the solution satisfying the IC. is given by*

$$\mathbf{x}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t). \tag{10.11}$$

Proof. Since $W[\mathbf{x}^{(1)}(0), \cdots, \mathbf{x}^{(n)}(0)] = \det I = 1 \neq 0$ we see $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ are fundamental set of solutions. Clearly (10.11) satisfy IC. □

Let

$$X(t) = (\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)).$$

Then any solution satisfying the initial condition (10.8) is given by $\mathbf{x}(t) = X(t)\mathbf{k}$.

Nonhomogeneous System

If $\mathbf{x}_p$ is a particular solution of nonhomogeneous system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \tag{10.12}$$

then the general solution of (10.12) is given by

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p,$$

where $\mathbf{x}_c = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$ is the general solution of associated homogeneous system.

10.2 Homogeneous Linear System with constant coefficients

Here we will study how to find fundamental set of solutions.

First consider the DE.

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{x}.$$

The solutions are

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}.$$

Both solutions are has the form

$$\mathbf{x}_i = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{r_i t}.$$

We will see the solution is generally given in this form when the matrix A has constant coefficients.

Eigenvalues and Eigenvectors

Given $n \times n$ matrix A consider the DE

$$\mathbf{x}' = A\mathbf{x}. \tag{10.13}$$

For a vector $\mathbf{k} \in \mathbb{R}^n$ we assume

$$\boxed{\mathbf{x} = \mathbf{k}e^{rt}} \tag{10.14}$$

and substitute into (10.13) we obtain

$$r\mathbf{k}e^{rt} = A\mathbf{k}e^{rt}.$$

Dividing by e^{rt} we obtain

$$\boxed{A\mathbf{k} = r\mathbf{k}}.$$

From this we get

$$det(A - rI) = 0. \tag{10.15}$$

This is called the **characteristic equation.** Solving the eigenvalue problem we obtain the solution of $\mathbf{x} = \mathbf{k}e^{rt}$.

Depending on the roots(eigenvalues) of the characteristic equation, the solution methods are classified into the following cases:

(1) Real and distinct eigenvalues

(2) Repeated eigenvalues (real)

(3) Complex eigenvalues

10.2.1 Real and distinct

When the eigenvalues of A are real and distinct, then general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{k}^{(1)}e^{r_1 t} + c_2\mathbf{k}^{(2)}e^{r_2 t} + \cdots + c_3\mathbf{k}^{(n)}e^{r_n t}.$$

Example 10.2.1. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}.$$

Sol. The characteristic equation is

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \tag{10.16}$$

$$|A - rI| = \begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = r^2 + 3r + 2 = 0.$$

So $r_1 = -1, r_2 = -2$.

(1) Case $r_1 = -1$:

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.17}$$

So $k_1 - k_2 = 0$ and we can choose

$$\mathbf{k}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{10.18}$$

(2) Case $r = -2$:

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.19}$$

So $3k_1 - 2k_2 = 0$ and we can choose

$$\mathbf{k}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{10.20}$$

Finally we have

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}.$$

Example 10.2.2. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}.$$

The characteristic equation is

$$(A - rI)\mathbf{k} = \begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \qquad (10.21)$$

$$\begin{aligned} |A - rI| &= \begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} \\ &= -r^3 + 4r^2 + r - 4 = -(r-4)(r-1)(r+1) = 0. \end{aligned}$$

So $r_1 = 4, r_2 = 1, r_3 = -1$.

(1) $r = 4$:

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \qquad (10.22)$$

$$\begin{aligned} -3k_1 \quad +k_2 \quad +2k_3 &= 0 \\ k_1 \quad -2k_2 \quad +k_3 &= 0 \\ 2k_1 \quad +k_2 \quad -3k_3 &= 0. \end{aligned}$$

Choose $k_3 = 1$ so that

$$\begin{aligned} -3k_1 \quad +k_2 &= -2 \\ k_1 \quad -2k_2 &= -1 \\ 2k_1 \quad +k_2 &= 3 \end{aligned}$$

from which we obtain $k_1 = 1, k_2 = 1$, i.e.,

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}.$$

(2) $r = 1$:

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \qquad (10.23)$$

$$\begin{array}{rrrl} & k_2 & +2k_3 & = 0 \\ k_1 & +k_2 & +k_3 & = 0 \\ 2k_1 & +k_2 & & = 0. \end{array}$$

Choose $k_1 = 1$ so that

$$\begin{array}{lrl} k_2 & +2k_3 & = 0 \\ k_2 & +k_3 & = -1 \\ k_2 & & = -2 \end{array}$$

from which $k_2 = -2, k_3 = 1$, i.e.,

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^t.$$

(3) $r = -1$:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{10.24}$$

$$\begin{array}{rrrl} 2k_1 & +k_2 & +2k_3 & = 0 \\ k_1 & +3k_2 & +k_3 & = 0 \\ 2k_1 & +k_2 & +2k_3 & = 0. \end{array}$$

Choose $k_3 = 1$ then

$$\begin{array}{rrl} 2k_1 & +k_2 & = -2 \\ k_1 & +3k_2 & = -1 \\ 2k_1 & +k_2 & = -2 \end{array}$$

from which $k_1 = -1, k_2 = 0$, i.e.,

$$\mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

Hence the general solution is

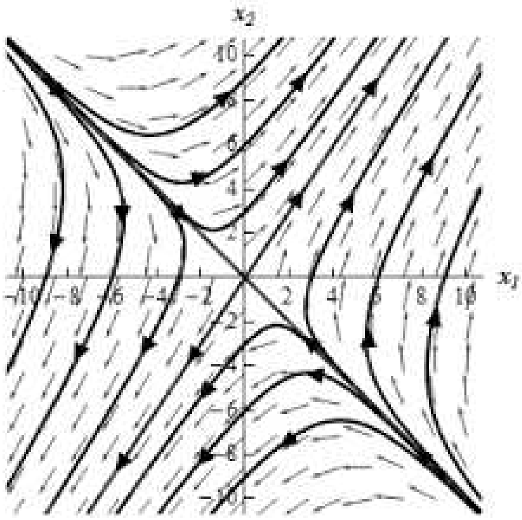
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^t + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

Remark 10.2.3. In this example A is symmetric, in which case it is known that there always exist n linearly independent vectors. So finding the solution is simple.

Phase portrait or Phase plane

Example 10.2.4.

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$



Sol. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 2-r & 3 \\ 2 & 1-r \end{vmatrix} = (r+1)(r-4) = 0,\ r_1 = -1, r_2 = 4.$$

For $r = -1$ the eigenvector is $\mathbf{k}_1 = (1, -1)^T$. For $r = 4$ the eigenvector is $\mathbf{k}_2 = (3, 2)^T$. So the solution of DE. is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

If we eliminate parameter t and get relation between x and y, (use various constants) then we get certain relations. For example, if $c_1 = 1, c_2 = 0$, we get $x(t) = e^{-t}, y(t) = -e^{-t}$, hence $y = -x$. If $c_1 = 0, c_2 = 1$, we get $x(t) = 3e^{4t}, y(t) = 2e^{4t}$ and hence $y = \frac{2}{3}x$. These solutions corresponds to the two blue lines.

Exercise 10.2.5. (1) Find the solution of the following DE.

(a)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

(b)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

(c)

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{x}$$

(d)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(e)

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

10.2.2 Repeated eigenvalues of multiplicity m

Assume r is a repeated eigenvalue of multiplicity m. There are two cases:

- There exists m linearly independent eigenvectors $\mathbf{k}^{(1)}, \cdots, \mathbf{k}^{(m)}$ corresponding to the eigenvalue r. In this case, the m-linearly independent solutions are given by
$$c_1\mathbf{k}^{(1)}e^{r_1 t} + \cdots + c_m\mathbf{k}^{(m)}e^{r_m t}$$

- There exists only one linearly independent eigenvector $\mathbf{k}^{(1)}$ corresponding to the eigenvalue r. In this case, the m-linearly independent solutions are given by (Solve the system in this order)
$$\begin{aligned} \mathbf{x}_1 &= \mathbf{k}^{(1)}e^{r_1 t} \\ \mathbf{x}_2 &= \mathbf{k}^{(1)}te^{r_1 t} + \mathbf{k}^{(2)}e^{r_1 t} \\ \mathbf{x}_2 &= \mathbf{k}^{(1)}\frac{t^2}{2!}e^{r_1 t} + \mathbf{k}^{(2)}te^{r_1 t} + \mathbf{k}^{(3)}e^{r_1 t} \\ &= \cdots \\ \mathbf{x}_m &= \mathbf{k}^{(1)}\frac{t^{m-1}}{(m-1)!}e^{r_1 t} + \mathbf{k}^{(2)}\frac{t^{m-2}}{(m-2)!}e^{r_1 t} + \cdots + \mathbf{k}^{(m)}e^{r_1 t}. \end{aligned}$$

  Vectors $\mathbf{k}^{(1)}, \mathbf{k}^{(2)}$ through $\mathbf{k}^{(m)}$ are obtained by substituting these expressions into the D.E.

Example 10.2.6. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}. \tag{10.25}$$

Sol. The characteristic equation is

$$\begin{vmatrix} 1-r & -2 & 2 \\ -2 & 1-r & -2 \\ 2 & -2 & 1-r \end{vmatrix} = -(r+1)^2(r-5) = 0. \tag{10.26}$$

For $r = -1$

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \mathbf{k}^{(1)} = \mathbf{0}.$$

Thus we have $k_1 - k_2 + k_3 = 0$. The two independent solution vectors are $\mathbf{k}^{(1)} = (1,1,0)^T$ and $\mathbf{k}^{(2)} = (0,1,1)^T$. For $r = 5$,

$$\begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix} \mathbf{k}^{(3)} = \mathbf{0}.$$

So $\mathbf{k}^{(3)} = (1,-1,1)^T$. In this case, there are three independent vectors. Hence the general solution is of the form

$$\mathbf{x}(t) = c_1\mathbf{k}^{(1)}e^{-t} + c_2\mathbf{k}^{(2)}e^{-t} + c_3\mathbf{k}^{(3)}e^{5t}.$$

Less than m - Linearly independent eigenvectors - Second solution

When r is a multiple eigenvalue of multiplicity 2 and if there is only one eigenvector corresponding to it then the first solution is given by as before,

$$\mathbf{x}^{(1)} = \mathbf{k}e^{rt}, \tag{10.27}$$

where $\mathbf{k}$ satisfies

$$(A - rI)\mathbf{k} = 0. \tag{10.28}$$

The second solution is

$$\mathbf{x}^{(2)} = \mathbf{k}te^{rt} + \mathbf{p}e^{rt}, \tag{10.29}$$

where the vector $\mathbf{p}$ can be found by

$$(A - rI)\mathbf{p} = \mathbf{k}. \tag{10.30}$$

The final solution is

$$\mathbf{x} = c_1\mathbf{k}e^{rt} + c_2(\mathbf{k}te^{rt} + \mathbf{p}e^{rt}).$$

Example 10.2.7. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \mathbf{x}. \tag{10.31}$$

Sol. The characteristic equation is

$$\begin{pmatrix} 3-r & -1 \\ 1 & 5-r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.32}$$

$$|A - rI| = \begin{vmatrix} 3-r & -1 \\ 1 & 5-r \end{vmatrix} = (r-4)^2 = 0.$$

So $r = r_1 = r_2 = 4$ and the equation to for the eigenvectors is:

$$\begin{aligned} -k_1 \quad -k_2 \quad &= 0 \\ k_1 \quad +k_2 \quad &= 0. \end{aligned}$$

Solving it, we get $k_1 = 1, k_2 = -1$. Hence we have only one linearly independent vector:

$$\mathbf{k} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from which we get one solution:

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}.$$

We need to find another linearly independent solution. Recall scalar case, we tried: $x(t) = c_1 e^{rt} + c_2 t e^{rt}$. So we may try a solution like $\mathbf{k} t e^{4t}$, but this is not enough! We have to add a term corresponding to the derivative of $\mathbf{k} t e^{4t}$. Thus try

$$\boxed{\mathbf{x}^{(2)} = \mathbf{k} t e^{4t} + \mathbf{p} e^{4t}}. \tag{10.33}$$

Substitute this into the DE., we get

$$(A - 4I)\mathbf{p} = \mathbf{k} \tag{10.34}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{10.35}$$

So we obtain $p_1 + p_2 = -1$. Set $\eta_1 = k$ then $p_2 = -1 - k$ and we obtain

$$\mathbf{p} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the second term (in red) is absorbed into **k** (so into the first solution $\mathbf{x}^{(1)}$), we can set

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t}.$$

So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t} \right]$$

Example 10.2.8. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{x}. \tag{10.36}$$

Sol. The characteristic equation is $(3-r)(-9-r)+36 = (r+3)^2 = 0$. The eigenvector are found from

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.37}$$

We get one eigenvector $\mathbf{k} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Hence $\mathbf{x}^{(1)} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$. For the second solution, we set

$$\mathbf{x}^{(2)} = \mathbf{k}te^{-3t} + \mathbf{p}e^{-3t}. \tag{10.38}$$

Substitute into DE., we see

$$(\mathbf{k}(1-3t) - 3\mathbf{p})e^{-3t} = (A\mathbf{k}t + A\mathbf{p})e^{-3t}.$$

Comparing, we get

$$(A+3I)\mathbf{k} = 0, \quad (A+3I)\mathbf{p} = \mathbf{k} = (3,1)^T.$$

$$(A+3I)\mathbf{p} = \mathbf{k} \Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \tag{10.39}$$

So $2p_1 - 6p_2 = 1$. We have has many solutions. Set p_2 free so that

$$\begin{pmatrix} 3p_2 + \frac{1}{2} \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

As before, we can set $p_2 = 0$ to get $\mathbf{p} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, thus

$$\mathbf{x}^{(2)} = \mathbf{k}te^{-3t} + \mathbf{p}e^{-3t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}.$$

Hence the final solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right].$$

Multiplicity 3 - Third solution

Similar method works when the multiplicity is higher, say $m = 3, 4$ etc. Assume r is a multiple eigenvalue of multiplicity 3 and there is only one eigenvector corresponding to it. Then the first and the second solution are given in the form (10.27), (10.29), i.e., the first solution is

$$\mathbf{x}^{(1)} = \mathbf{k}e^{rt}, \tag{10.40}$$

where $\mathbf{k}$ satisfies

$$(A - rI)\mathbf{k} = 0. \tag{10.41}$$

The second solution is

$$\mathbf{x}^{(2)} = \mathbf{k}te^{rt} + \mathbf{p}e^{rt}, \tag{10.42}$$

where the vector $\mathbf{p}$ can be found by

$$(A - rI)\mathbf{p} = \mathbf{k}. \tag{10.43}$$

Finally, the third solution is given by

$$\mathbf{x}^{(3)} = \mathbf{k}\frac{t^2}{2}e^{rt} + \mathbf{p}te^{rt} + \mathbf{q}e^{rt}, \tag{10.44}$$

where the vectors $\mathbf{k}, \mathbf{p}$ can be found as follows:

$$(A - rI)\mathbf{k} = 0 \tag{10.45}$$

$$(A - rI)\mathbf{p} = \mathbf{k} \tag{10.46}$$

$$(A - rI)\mathbf{q} = \mathbf{p}. \tag{10.47}$$

Example 10.2.9. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} \tag{10.48}$$

Sol. The characteristic equation is $(r-2)^3 = 0$ so $r = 2$ is a triple root and we have $(A - 2I)\mathbf{k} = 0$,

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$k_2 + 6k_3 = 0, \;\; 5k_3 = 0 \Rightarrow k_2 = k_3 = 0$$

and we obtain one independent eigenvector: $\mathbf{k} = (1, 0, 0)^T$. The first solution is

$$\mathbf{x}^{(1)} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

The second solution can be found by solving $(A - 2I)\mathbf{p} = \mathbf{k}$.

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solving we see $p_2 + 6p_3 = 1, \;\; 5p_3 = 0 \Rightarrow p_3 = 0, p_2 = 1, p_1$ is free. So we get

$$\mathbf{p} = p_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since the first vector is included in $\mathbf{k}$, we choose $p_1 = 0$. Hence

$$\mathbf{x}^{(2)} = \mathbf{k}te^{rt} + \mathbf{p}e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + p_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

Finally for the third, we solve $(A - 2I)\mathbf{q} = \mathbf{p} = (0, 1, 0)^T$, i.e.,

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{q} = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}.$$

So the general solution is

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{k}e^{rt} + c_2[\mathbf{k}te^{rt} + \mathbf{p}e^{rt}] + c_3[\mathbf{k}\frac{t^2}{2}e^{rt} + \mathbf{p}te^{rt} + \mathbf{q}e^{rt}] \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] e^{2t} + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix} \right] e^{2t}. \end{aligned}$$

Exercise 10.2.10. (1) Find the solution of DE.

(a) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$

(c) $\mathbf{x}' = \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

(d) $\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{3}{2} \end{pmatrix} \mathbf{x}$

(e) $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

(f) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}$

(g) $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

(2) Solve the IVP:

(a) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(c) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}, \ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(3) Find the general solution of

$$\mathbf{x}' = A\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}$$

10.2.3 Complex roots

Assume the characteristic equation of

$$\mathbf{x}' = A\mathbf{x} \tag{10.49}$$

has two complex conjugate roots $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$ with the corresponding eigenvectors $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)}$. The solution in this case is

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1\mathbf{k}^{(1)}e^{r_1 t} + c_2\mathbf{k}^{(2)}e^{r_2 t},$$

Since A is real, the eigenvectors corresponding to r_1, r_2 are two complex conjugates vectors $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)} = \bar{\mathbf{k}}^{(1)}$. Set $\mathbf{k}^{(1)} = \mathbf{a} + i\mathbf{b}, \mathbf{k}^{(2)} = \mathbf{a} - i\mathbf{b}$.

Since

$$\begin{aligned} \mathbf{x}^{(1)} &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t}(\mathbf{a}\cos \mu t - \mathbf{b}\sin \mu t) + ie^{\lambda t}(\mathbf{a}\sin \mu t + \mathbf{b}\cos \mu t), \end{aligned}$$

$$\begin{aligned}\mathbf{x}^{(2)} &= (\mathbf{a} - i\mathbf{b})e^{(\lambda - i\mu)t} \\ &= (\mathbf{a} - i\mathbf{b})e^{\lambda t}(\cos\mu t - i\sin\mu t) \\ &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) - ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t),\end{aligned}$$

we see

$$\begin{aligned}\mathbf{u} &= \tfrac{\mathbf{x}^{(1)}+\mathbf{x}^{(2)}}{2} &&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} &= \tfrac{\mathbf{x}^{(1)}-\mathbf{x}^{(2)}}{2i} &&= e^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t)\end{aligned}$$

are linearly independent. So we may write

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} = c_1 e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + c_2 e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t),$$

where $\mathbf{a}$ is the real part and $\mathbf{b}$ is the imaginary part of $\mathbf{k}^{(1)}$ respectively.

Example 10.2.11. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\mathbf{x}$.

Solution. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 1-r & 3 \\ -3 & 1-r \end{vmatrix} = r^2 - 2r + 10 = 0$$

from which we obtain $r = 1 \pm 3i$. When $r_1 = 1 + 3i$

$$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.50}$$

We can choose eigenvectors

$$\mathbf{k}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{10.51}$$

and the second vector is $\mathbf{k}^{(2)} = \overline{\mathbf{k}^{(1)}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Hence

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+3i)t}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-3i)t}$$

or

$$\mathbf{u} = \frac{\mathbf{x}^{(1)} + \mathbf{x}^{(2)}}{2} = e^t\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}, \qquad \mathbf{v} = \frac{\mathbf{x}^{(1)} - \mathbf{x}^{(2)}}{2i} = e^t\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^t\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

10.3 Diagonalization

In this section, consider an alternative method to find solutions. Assume we have a system of DE:

$$\mathbf{x}' = A\mathbf{x} \tag{10.52}$$

where an $n \times n$ matrix A has n-linearly independent eigenvectors corresponding to $\lambda_1, \cdots, \lambda_n$, i.e.,

$$A\mathbf{k}^{(i)} = \lambda_i \mathbf{k}^{(i)}.$$

Let P be the matrix P whose columns consist of eigenvectors of A. Then using the matrix P we can diagonalize the system:

Let $P = (\mathbf{k}^{(1)}, \cdots, \mathbf{k}^{(n)})$. Then we have such that $P^{-1}AP = D$ where $D = diag(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix. Then with the substitution $\mathbf{x} = P\mathbf{y}$ we have

$$(P\mathbf{y})' = AP\mathbf{y} \Leftrightarrow \mathbf{y}' = P^{-1}AP\mathbf{y} = D\mathbf{y}.$$

The last equation is easy to solve:

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and the solution is $y_1 = c_1 e^{\lambda_1 t},\ y_2 = c_2 e^{\lambda_2 t}, \cdots,\ y_n = c_n e^{\lambda_n t}$. Hence we have

$$\mathbf{x} = P\mathbf{y} = \left(\mathbf{k}^{(1)}, \cdots, \mathbf{k}^{(n)}\right) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t}. \end{pmatrix} = c_1 \mathbf{k}^{(1)} e^{\lambda_1 t} + \cdots + c_n \mathbf{k}^{(n)} e^{\lambda_n t}. \tag{10.53}$$

Example 10.3.1. Solve 2×2 system of DE.

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$$

Solution. The charac. eq. is

$$\det(A - \lambda I) = 0.$$

From this we have

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2 - 1 = 0, \quad \lambda = 1, 3.$$

When $\lambda = 1$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

the eigenvector is

$$\mathbf{k}_1 = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$

$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

the eigenvector is

$$\mathbf{k}_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $\mathbf{k}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{k}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then with $P = (\mathbf{k}_1, \mathbf{k}_2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we have

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = D.$$

Thus

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Example 10.3.2 (p. 570)**.** Solve the system of DE.

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} \mathbf{x}.$$

From $det(A - rI) = 0$, we get $-(2+r)((r+3)(r-9)+32) = -(2+r)(r-1)(r-5) = 0$. Hence $r = -2, 1, 5$. Eigenvectors are

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{k}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{k}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P^{-1}AP = D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

The solution of the diagonal system $\mathbf{y}' = D\mathbf{y}$ is $\mathbf{y} = \left(c_1e^{-2t}, c_2e^t, c_3e^{5t}\right)^T$. Hence

$$\mathbf{x} = P\mathbf{y} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{-2t} \\ c_2e^t \\ c_3e^{5t} \end{pmatrix} = \begin{pmatrix} c_1e^{-2t} + 2c_2e^t + c_3e^{5t} \\ 2c_2e^t + c_3e^{5t} \\ c_2e^t + c_3e^{5t} \end{pmatrix}. \tag{10.54}$$

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t}.$$

Exercise 10.3.3. (1) Find the solution of

(a) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

(b) $\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}\mathbf{x}$

(c) $\mathbf{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 3 \end{pmatrix}\mathbf{x}$

(d) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}\mathbf{x}$

(e) $\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{3}{2} \end{pmatrix}\mathbf{x}$

(f) $\mathbf{x}' = \begin{pmatrix} 3 & -8 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

(g) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{x}$

(h) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix}\mathbf{x}$

10.4 Nonhomogeneous Linear Systems

We now study how to solve nonhomogeneous linear system of DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{10.55}$$

As in the case of single DE. we separate the homogeneous case $\mathbf{x}' = A\mathbf{x}$ and the solution will be given by

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p,$$

where $\mathbf{x}_h$ is the solution of the homogeneous problem and $\mathbf{x}_p$ is a particular solution of the nonhomogeneous problem.

10.4.1 Method of Undetermined Coefficients

This works only when the coefficients of A are constant case, and right hand side terms are **constants, polynomials, exponential functions, sines, cosines or finite linear combinations of such functions**!

Example 10.4.1. Solve $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$.

First the charac. eq. of homogeneous equation is

$$\begin{vmatrix} -1-r & 2 \\ -1 & 1-r \end{vmatrix} = r^2+1=0.$$

And the eigenvectors corresponding to $r = i, -i$ are $(1-i,1)^T$ and $(1+i,1)^T$. Hence

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{-it}$$

or take real part and imaginary part of

$$\begin{aligned}(\cos t + i\sin t)\begin{pmatrix} 1-i \\ 1 \end{pmatrix} &= \begin{pmatrix} (1-i)\cos t + (i+1)\sin t \\ \cos t + i\sin t \end{pmatrix} \\ &= \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.\end{aligned}$$

we get

$$c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Particular sol. Since $\mathbf{f}(t)$ is constant, we let $\mathbf{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and find

$$0 = A\mathbf{x}_p + \mathbf{f} = \begin{pmatrix} -a_1 + 2b_1 - 8 \\ -a_1 + b_1 + 3 \end{pmatrix}.$$

So $\mathbf{x}_p = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$.

Example 10.4.2 (nonconstant rhs)**.** Solve $\mathbf{x}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$.

Eigenvalues are $r_1 = 2, r_2 = 7$ and the eigenvectors are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence the complementary solution is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}.$$

For a **particular solution**, let

$$\mathbf{x}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

and substitute into the DE and find the numbers a_1, b_1, a_2, b_2.

$$\begin{aligned}\begin{pmatrix}a_2\\b_2\end{pmatrix} &= \begin{pmatrix}6&1\\4&3\end{pmatrix}\left[\begin{pmatrix}a_2\\b_2\end{pmatrix}t+\begin{pmatrix}a_1\\b_1\end{pmatrix}\right]+\begin{pmatrix}6\\-10\end{pmatrix}t+\begin{pmatrix}0\\4\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} &= \begin{pmatrix}(6a_2+b_2+6)t+6a_1+b_1-a_2\\(4a_2+3b_2-10)t+4a_1+3b_1-b_2+4\end{pmatrix}\end{aligned}$$

Hence

$$\begin{pmatrix}6a_2+b_2+6 &= 0\\4a_2+3b_2-10 &= 0\end{pmatrix} \text{ and } \begin{pmatrix}6a_1+b_1-a_2 &= 0\\4a_1+3b_1-b_2+4 &= 0\end{pmatrix}$$

Solving first set of eqs we get $a_2 = -2, b_2 = 6$. We then substitute it into the second set of eqs to get $a_1 = -\frac{4}{7}, b_1 = \frac{10}{7}$. Therefore

$$\mathbf{x}_p = \begin{pmatrix}-2\\6\end{pmatrix}t+\begin{pmatrix}-\frac{4}{7}\\\frac{10}{7}\end{pmatrix}.$$

and the general solution of DE is

$$\mathbf{x} = c_1\begin{pmatrix}1\\-4\end{pmatrix}e^{2t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{7t}+\begin{pmatrix}-2\\6\end{pmatrix}t+\begin{pmatrix}-\frac{4}{7}\\\frac{10}{7}\end{pmatrix}.$$

Example 10.4.3 (nonconstant rhs 2)**.** Solve

$$\begin{aligned}\frac{dx}{dt} &= 5x+3y-2e^{-t}+1\\ \frac{dy}{dt} &= -x+y+e^{-t}-5t+7.\end{aligned}$$

The rhs can be written as

$$\mathbf{F}(t) = \begin{pmatrix}-2\\1\end{pmatrix}e^{-t}+\begin{pmatrix}0\\-5\end{pmatrix}t+\begin{pmatrix}1\\7\end{pmatrix}.$$

Hence we try

$$\mathbf{x}_p = \begin{pmatrix}a_1\\b_1\end{pmatrix}e^{-t}+\begin{pmatrix}a_2\\b_2\end{pmatrix}t+\begin{pmatrix}a_1\\b_1\end{pmatrix}.$$

Notice the difference in the candidates. Generally, we had better use the next method.

10.4.2 Variation of Parameters

A Fundamental matrix - Homogeneous system

If $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are fundamental set of solutions of homog. system $\mathbf{x}' = A\mathbf{x}$, then the general solution of homog. system is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$, or in matrix form

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{c}, \tag{10.56}$$

where $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$, and $\mathbf{\Phi}(t)$ is the matrix whose columns are vectors $\mathbf{x}_i, i = 1, 2, \cdots, n$:

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

called a **fundamental matrix**. We note that

- The fundamental matrix $\mathbf{\Phi}(t)$ is nonsingular
- If $\mathbf{\Phi}(t)$ is a fundamental matrix of the system $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{\Phi}'(t) = A\mathbf{\Phi}(t). \tag{10.57}$$

Variation of Parameters - Nonhomogeneous system

To find a particular solution we may use the technique of section 3.5. i.e., replace the constant coefficient $\mathbf{c}$ by functions

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \tag{10.58}$$

so that $\mathbf{x}_p = \mathbf{\Phi}(t)\mathbf{u}(t)$ is a particular solution of the system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}. \tag{10.59}$$

Taking derivative we obtain

$$\mathbf{x}_p' = \mathbf{\Phi}(t)\mathbf{u}'(t) + \mathbf{\Phi}'(t)\mathbf{u}(t). \tag{10.60}$$

Substitute it into (10.59)

$$\mathbf{\Phi}(t)\mathbf{u}'(t) + \mathbf{\Phi}'(t)\mathbf{u}(t) = A\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{f}(t). \tag{10.61}$$

Since $\mathbf{\Phi}'(t) = A\mathbf{\Phi}(t)$ we have

$$\mathbf{\Phi}(t)\mathbf{u}'(t) = \mathbf{f}(t). \tag{10.62}$$

$$\mathbf{u}'(t) = \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) \;\Rightarrow\; \mathbf{u}(t) = \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt.$$

Since $\mathbf{x}_p = \mathbf{\Phi}(t)\mathbf{u}(t)$ we have

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt. \tag{10.63}$$

Hence the general solution of the system is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt. \tag{10.64}$$

Example 10.4.4. Solve the DE.

$$\mathbf{x} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}. \tag{10.65}$$

The charac. equation is

$$det(A - rI) = \begin{vmatrix} -3-r & 1 \\ 2 & -4-r \end{vmatrix} = (r+2)(r+5) = 0.$$

Eigenvectors corresponding to $r = -2, r = -5$ are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The solution of homog. system is

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}.$$

The fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \text{ and } \mathbf{\Phi}(t)^{-1} = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}.$$

Hence by (10.63)

$$\begin{aligned}
\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) &= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \int \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix} \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \int \begin{pmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \begin{pmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}
\end{aligned}$$

Hence the solution of the nonhomg system is

$$\mathbf{x}(t) \quad = \quad c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t} + \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Initial Value Problems

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)ds. \tag{10.66}$$

If the solution is to satisfy IC $\mathbf{x}(t_0) = \mathbf{x}_0$ then we must have $\mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{c}$, so

$$\mathbf{c} = \mathbf{\Phi}(t_0)^{-1}\mathbf{x}(t_0).$$

Hence the solution of IVP is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(t_0)^{-1}\mathbf{x}(t_0) + \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)ds. \tag{10.67}$$

10.4.3 Nonhomogeneous Problem by Diagonalization

We assume A is diagonalizable. In other words, there exists a matrix P such that

$$P^{-1}AP = D \text{ is diagonal}$$

Substituting $\mathbf{x} = P\mathbf{y}$ into $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we get

$$P\mathbf{y}' = AP\mathbf{y} + \mathbf{f} \text{ or } \mathbf{y}' = P^{-1}AP\mathbf{y} + P^{-1}\mathbf{f} = D\mathbf{y} + P^{-1}\mathbf{f}.$$

Example 10.4.5. Solve the DE.

$$\mathbf{x} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}. \tag{10.68}$$

The charac. equation is

$$det(A - rI) = \begin{vmatrix} 4-r & 2 \\ 2 & 1-r \end{vmatrix} = r(r-5) = 0.$$

Eigenvectors corresponding to $r = 0, r = 5$ are

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. Using $\mathbf{x} = P\mathbf{y}$ and

$$P^{-1}\mathbf{f} = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3e^t \\ e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{pmatrix},$$

we get the uncoupled system

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{pmatrix}.$$

Thus

$$y_1' = \frac{1}{5}e^t \text{ and } y_2' = 5y_2 + \frac{7}{5}e^t.$$

Solving for $\mathbf{y}$ we get

$$y_1 = \frac{1}{5}e^t + c_1 \quad \text{and} \quad -\frac{7}{20}e^t + c_2 e^{5t}.$$

Hence the solution is

$$\begin{aligned} \mathbf{x} &= P\mathbf{y} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}e^t + c_1 \\ -\frac{7}{20}e^t + c_2 e^{5t} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}e^t + c_1 + 2c_2 e^{5t} \\ -\frac{3}{4}e^t - 2c_1 + c_2 e^{5t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} e^t. \end{aligned}$$

10.5 Matrix exponential

To solve a system of linear ODE with constant coefficient ($\mathbf{x}' = A\mathbf{x}$), we can use a method similar to scalar DE, i,e., setting $\mathbf{x} = e^{At}$, where the exponential of a matrix has to be properly understood. We recall

$$e^{at} = 1 + at + a^2\frac{t^2}{2!} + \cdots a^m\frac{t^m}{m!} + \cdots = \sum_{k=0}^{\infty} a^k \frac{t^k}{k!}$$

We similarly define, for a matrix A :

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^m\frac{t^m}{m!} + \cdots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}. \tag{10.69}$$

Example 10.5.1. Find the e^{At} when

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{10.70}$$

Sol.

$$A^2 = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix},\ A^3 = \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix},\ A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

$$\begin{aligned} e^{At} &= I + At + A^2\frac{t^2}{2!} + \cdots + \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} t + \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix}\frac{t^3}{3!} + \cdots \\ &= \begin{pmatrix} 1 + 2t + 2^2\frac{t^2}{2!} + \cdots & 0 \\ 0 & 1 + 3t + 3^2\frac{t^2}{2!} + \cdots \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \end{aligned}$$

Thus for a $n \times n$ diagonal matrix A with diagonal entries $a_1, a_2, \cdots, a_n$, we see

$$e^{At} = \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 & 0 \\ 0 & e^{a_2 t} & 0 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & e^{a_n t} \end{pmatrix}$$

Derivatives of e^{At}

The derivatives of a matrix function can be computed as

$$\frac{d}{dt}e^{At} = Ae^{At}. \tag{10.71}$$

Use the series expansion (10.69).

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt}\left[I + At + A^2\tfrac{t^2}{2!} + \cdots + A^m\tfrac{t^m}{m!} + \cdots\right] \\ &= A + A^2 t + A^3\frac{t^2}{2!} + \cdots \\ &= A\left[I + At + A^2\tfrac{t^2}{2!} + \cdots\right] = Ae^{At}. \end{aligned}$$

We can show the general solution of the DE $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = e^{At}\mathbf{C}$ for some constant vector $\mathbf{C}$ since

$$\mathbf{x}' = \frac{d}{dt}e^{At}\mathbf{C} = Ae^{At}\mathbf{C} = A\mathbf{x}. \tag{10.72}$$

e^{At} is a fundamental matrix

Let us write $\mathbf{\Phi}(t) = e^{At}$. Then we see $\mathbf{\Phi}'(t) = A\mathbf{\Phi}$ and $\mathbf{\Phi}(0) = e^{A0} = I$ and $\mathbf{\Phi}(0) \neq 0$ thus $\mathbf{\Phi}(t)$ is a fundamental set of solutions, or a fundamental matrix.

Hence any solution of homog. system $\mathbf{x}' = A\mathbf{x}$ is given by $e^{At}\mathbf{C}$ for some vector $\mathbf{C}$.

Nonhomog. systems

In view of techniques studied for scalar equations we can see the solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$ is given by

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = e^{At}\mathbf{C} + e^{At}\int_{t_0}^{t} e^{-As}\mathbf{F}(s)ds. \tag{10.73}$$

Here e^{-As} is the matrix inverse of e^{As} and obtained by replacing s by $-s$.

Laplace transform

Let us recall $\mathbf{X}(t) = e^{At}$ is the fundamental set of sols. satisfying the IC, i.e.

$$\mathbf{X}' = A\mathbf{X},\ \mathbf{X}(0) = I. \tag{10.74}$$

Use Laplace transform. If $\mathbf{x}(s) = \mathcal{L}\{\mathbf{X}(t)\} = \mathcal{L}\{e^{At}\}$, then we see

$$s\mathbf{x}(s) - \mathbf{X}(0) = A\mathbf{x}(s) \text{ or } (sI - A)\mathbf{x}(s) = I.$$

We have used small capital for transformed function and large capital for original function. Multiplying its inverse, we see

$$\mathbf{x}(s) = (sI - A)^{-1}I = (sI - A)^{-1}.$$

In other words, $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$ or

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}. \tag{10.75}$$

Compare this with the formula:

$$e^{at} = \mathcal{L}^{-1}\{\frac{1}{(s-a)}\}.$$

This result can be used to find a matrix exponential.

Example 10.5.2. Use Laplace Transform to find e^{At} when

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}. \tag{10.76}$$

In general a direction evaluation of e^{At} is very complicated. However, if we use Laplace Transform of e^{At} and do some algebraic manipulation on s-space, then use inverse Laplace Transform, we sometimes compute e^{At} easily.

Sol. First recall $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ and so

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1} \text{ or } e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}. \tag{10.77}$$

We will compute $(sI - A)^{-1}$ first. Since

$$sI - A = \begin{pmatrix} s-1 & 1 \\ -2 & s+2 \end{pmatrix},$$

we have

$$(sI - A)^{-1} = \begin{pmatrix} s-1 & 1 \\ -2 & s+2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{s+2}{s(s+1)} & \frac{-1}{s(s+1)} \\ \frac{2}{s(s+1)} & \frac{s-1}{s(s+1)} \end{pmatrix}.$$

Decomposing the entries we see

$$(sI - A)^{-1} = \begin{pmatrix} \frac{2}{s} - \frac{1}{s+1} & -\frac{1}{s} + \frac{1}{s+1} \\ \frac{2}{s} - \frac{2}{s+1} & -\frac{1}{s} + \frac{2}{s+1} \end{pmatrix}.$$

Taking the inverse Laplace Transform, we get by (10.77)

$$e^{At} = \begin{pmatrix} 2 - e^{-t} & -1 + e^{-t} \\ 2 - 2e^{-t} & -1 + 2e^{-t} \end{pmatrix}.$$

□